ReLU Version of Dense Networks

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Abstract

This study investigates how replacing the sine coupling function with the Rectified Linear Unit (ReLU), defined as $\max\{0, \sin(x)\}$, affects synchronization in Kuramoto oscillator networks. We analyze both identical and non-identical oscillator scenarios across dense networks and two-group social networks exhibiting internal competition and external attraction. Our findings demonstrate that ReLU-based coupling significantly enhances synchronization, relying primarily on network connectivity rather than on degree distribution or frequency similarity, in sharp contrast to the classical Kuramoto model. We provide sufficient theoretical conditions for achieving synchronization, validated by numerical simulations, emphasizing the robustness of the ReLU modification against structural and frequency variations.

1 Introduction

The phenomenon of synchronization is widespread in the natural world. For example, in biological systems, synchronization is observed in the coordinated chirping of crickets and the synchronous flashing of fireflies. Beyond biology, synchronization is evident in the swinging of pendulums, the stability of electrical grids, and the consistency of certain chemical reactions [2, 4, 33].

Synchronization is not only of interest from a biological or physical perspective but is also crucial in various technological applications. In electrical engineering, the stability of power grids relies on the synchronization of generators [13]. In neuroscience, the coordinated firing of neurons is essential for proper brain function and for the rhythmic beating of the heart [5, 14]. In chemistry, synchronization can give rise to oscillatory reactions that are fundamental to certain biochemical processes [11]. Consequently, the study of synchronization spans multiple disciplines, each providing unique insights into this complex phenomenon.

Among the mathematical models that describe synchronous behavior, the Kuramoto model, proposed by Kuramoto in 1975, has received significant attention [25]. The classical Kuramoto model is given by

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{\ell=1}^{N} \sin\Big(\theta_\ell(t) - \theta_i(t)\Big), \quad \text{for } t > 0 \text{ and } i = 1, 2, \ldots, N, \tag{1.1}$$

where $\theta_i(t)$ is the phase of the ith oscillator, $\dot{\theta}_i(t) = \frac{d\theta_i(t)}{dt}$ represents its frequency, ω_i is its natural frequency, N is the number of oscillators, and K is the coupling strength. The interaction term $\sin\big(\theta_\ell(t) - \theta_i(t)\big)$ encapsulates the coupling between pairs of oscillators and is central to the emergence of synchronization.

The Kuramoto model is particularly interesting because it exhibits a phase transition from incoherence to synchronization as the coupling strength K increases. When the phases of two oscillators are close (i.e., the phase difference is less than π), the leading oscillator decelerates while the trailing oscillator accelerates, thereby facilitating synchronization. This mechanism can lead to a collective synchronized state in which all oscillators move in unison despite having different natural frequencies.

The Kuramoto model has been extensively studied in various contexts, including its applications to different network topologies [23, 29], the effects of time delays [19, 20], and the influence of noise [22]. To gain a deeper understanding of synchronization in more complex systems, several extensions and modifications of the original model have been proposed. These include investigations into dense networks [24, 28, 30, 35, 37, 41], the incorporation of heterogeneous natural frequencies [6, 7], the strong competition variant [21], weighted coupling [12], the two-group Kuramoto model [8, 15], and higher-dimensional oscillators [17], each contributing valuable insights into the dynamics of synchronization.

For further analysis, we define the phase vector

$$\Theta(t) = \big(\theta_1(t), \theta_2(t), \ldots, \theta_N(t)\big).$$

We now introduce several definitions to formalize the concepts of phase and frequency synchronization.

Definition 1.1 (Complete Frequency Synchronization)**.** Suppose $\Theta(t)$ is a solution to (1.1). Then, $\Theta(t)$ is said to achieve *complete frequency synchronization asymptotically* if, for any $1 \le i, j \le N$,

$$\lim_{t\to\infty} \left|\dot{\theta}_i(t) - \dot{\theta}_j(t)\right| = 0.$$

Definition 1.2 (Complete Phase Synchronization)**.** Suppose $\Theta(t)$ is a solution to (1.1). Then, $\Theta(t)$ is said to achieve *complete phase synchronization asymptotically* if, for any $1 \le i, j \le N$, there exists an integer n_{ij} such that

$$\lim_{t\to\infty} \left|\theta_i(t) - \theta_j(t) - 2\pi n_{ij}\right| = 0.$$

Remark 1.3. *Clearly, complete phase synchronization implies complete frequency synchronization; hence, complete phase synchronization is a stronger condition.*

Definition 1.4 (Diameter Function)**.** For any $X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, the diameter of X is defined as

$$D(X) := \max_{1\le i,j\le N} \big(x_i - x_j\big).$$

Observe that

$$\lim_{t\to\infty} D(\Theta(t)) = 0$$

is equivalent to

$$\lim_{t\to\infty} \left|\theta_i(t) - \theta_j(t)\right| = 0 \quad \text{for all } 1 \le i, j \le N.$$

Thus, to investigate phase synchronization, it suffices to analyze the asymptotic behavior of $D(\Theta(t))$. Similarly,

$$\lim_{t\to\infty} D\big(\dot{\Theta}(t)\big) = 0$$

is equivalent to

$$\lim_{t\to\infty} \left|\dot{\theta}_i(t) - \dot{\theta}_j(t)\right| = 0 \quad \text{for all } 1 \le i, j \le N.$$

Hence, to study frequency synchronization, it is sufficient to examine the asymptotic behavior of $D(\Theta(t))$ and $D\big(\dot{\Theta}(t)\big)$ (see [6–8, 15, 18–21]), an approach that we also adopt in this paper.

We first delve into the Kuramoto model on dense networks and explore its ReLU-modified version. Following this, we integrate these concepts to present the ReLU-modified Kuramoto model on dense networks and its application within social network models.

1.1 Homogeneous Kuramoto Model on Dense Networks

In 2012, Taylor [35] investigated the collective behavior of identical Kuramoto oscillators on a dense network, modeled by

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{\ell=1}^{N} A_{i\ell} \sin\Big(\theta_\ell(t) - \theta_i(t)\Big), \quad \text{for } t > 0 \text{ and } i = 1, 2, \ldots, N, \tag{1.2}$$

Under the homogeneous assumption, all oscillators share the same natural frequency (i.e., $\omega_i = \omega$ for all i). The connectivity of the network is represented by the adjacency symmetry matrix A, whose entries are defined as

$$A_{ij} = \begin{cases} 1 & \text{if oscillator } i \text{ is coupled to oscillator } j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

By performing the change of variables $\theta_i(t) \to \theta_i(t) + \omega t$, we may assume without loss of generality that $\omega = 0$. Moreover, by rescaling the time variable t, we can assume $K = 1$. Consequently, (1.2) simplifies to

$$\dot{\theta}_i(t) = \sum_{\ell=1}^{N} A_{i\ell} \sin\Big(\theta_\ell(t) - \theta_i(t)\Big), \quad \text{for } t > 0 \text{ and } i = 1, 2, \ldots, N, \tag{1.4}$$

Without loss of generality, we assume that $A_{ii} = 0$ for all i, implying no self-coupling among oscillators. Furthermore, it is assumed that every vertex is connected to at least $\mu(N-1)$ other vertices, where N is the total number of oscillators. We define the critical connectivity μ_c as the smallest value of μ for which any network of N oscillators is globally synchronizing when $\mu \geq \mu_c$. Conversely, for any $\mu < \mu_c$, there exists at least one network configuration that may exhibit attractors other than the in-phase synchronizing state. According to Taylor's result [35],

$$\mu_c \leq 0.93.$$

This indicates that a sufficiently high level of connectivity guarantees global synchronization across the network.

Subsequently, Ling et al. [28] combined a Lyapunov function approach with nonconvex optimization methods to improve the upper bound of μ_c to below 0.7929, while Lu et al. [30] further reduced this upper bound to 0.7889 using similar techniques. Kassabov et al. [24] later refined this bound to 0.75, which is currently the best-known upper bound.

Regarding the lower bounds, Townsend et al. [37] proved in 2020 that $\mu_c > 0.6828$ and observed that at $\mu = 0.75$ the coupling system exhibits notable spectral properties, suggesting that the critical connectivity might be exactly 0.75. Yoneda et al. [41] improved this lower bound to $\mu_c > 0.6838$ in 2021. Nonetheless, the exact value of μ_c remains unknown.

For the case of non-identical oscillators, Ling [27] established the following result:

Theorem 1.5 (Ling [27]). *Consider the Kuramoto model (1.2) with natural frequencies $\{\omega_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n \omega_i = 0$. If*

$$\max_{1 \leq i \leq n} |\omega_i| < K\sqrt{\mu - \frac{3}{4}} + (\mu - 1),$$

then there exists a unique frequency-synchronized solution.

Remark 1.6. *Since $\max_{1 \leq i \leq n} |\omega_i| \geq 0$, Theorem 1.5 implies that $\mu > \frac{3-\sqrt{2}}{2} \approx 0.7929$, indicating that a relatively high level of connectivity is required.*

In both the identical and non-identical cases, a relatively high level of connectivity is typically necessary to achieve synchronization. However, as will be demonstrated later, in the ReLU-modified version of the model the mere connectivity of the matrix A is sufficient to achieve synchronization in the identical case (see Theorem 1.10). Moreover, the relationship between Ω and μ can be considerably relaxed in the non-identical case (see Theorem 1.11).

Next, we explore the ReLU version of the Kuramoto model.

1.2 ReLU Version of the Kuramoto Model

In the ReLU version of the Kuramoto, we consider $\max\{0, \sin(x)\}$ instead of $\sin(x)$ in (1.1). That is,

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{\ell=1}^{N} \max\{0, \sin(\theta_\ell(t) - \theta_i(t))\}, \quad \text{for } t > 0 \text{ and } i = 1, 2, \ldots, N, \tag{1.5}$$

The intuition is that the oscillator in the front will not chase the one behind but will stay in place and wait. This implies a purely competitive coupling system.

This type of dynamic coupling has been explored in the studies of Yang et al. [39, 40] through numerical experiments. However, research on this variant of the Kuramoto model remains limited, with Hsia and Tsai [21] being among the few who have further investigated this aspect. They introduce a variation termed the Strong Competition (SC) Kuramoto Model, akin to the ReLU model.

The classical Kuramoto model's analysis often leverages the oddness of the sine function, allowing the use of tools like the Lyapunov function [10, 18, 38] or the order parameter [24, 27, 28]. However, the ReLU version lacks this symmetry, making such methods less straightforward. To address this challenge, Hsia and Tsai [21] demonstrate the existence of a well-order among the oscillators, implying that they eventually align in an order corresponding to their natural frequencies. This concept is integral to our approach, and we will follow a similar idea in our analysis (see Step 1 in Theorem 1.10 and Lemma 4.2). Their observation leads to the formulation of the following theorems:

Theorem 1.7 (Hsia and Tsai [21]). *Given $\Theta(t)$ as a solution to (1.5) with $D(\Theta(0)) < \pi$ and all ω_i are identical, then*

$$\lim_{t\to\infty} D(\Theta(t)) = 0,$$

implying that the oscillators achieve complete phase synchronization asymptotically.

Theorem 1.8 (Hsia and Tsai [21]). *Assume $D(\Omega) < NK \sin\alpha$ for some $\alpha \in (0, \pi/2)$. Let $\Theta(t)$ be a solution to (1.5) with $D(\Theta(0)) < \pi - \delta$, then*

$$\lim_{t\to\infty} \dot{\theta}_i(t) = \max\{\omega_1, \omega_2, \dots, \omega_N\}, \quad \textit{for } i = 1, 2, 3, \dots, N.$$

Remark 1.9. *(a) In Theorem 1.7, similar to Section 1.1, by changing variables, we may assume $K = 1$ and $\omega_i = 0$ for $i = 1, 2, \dots, N$.*

(b) In the language of networks, the conditions of the theorem imply that $A_{ij} = 1$ for all $1 \le i < j < N$, i.e., $\mu = 1$.

(c) Theorem 1.7 and Theorem 1.8 are special cases of our Theorem 1.10 and Theorem 1.11, respectively.

1.3 ReLU Version of the Kuramoto Model on Dense Network

We are now prepared to introduce our model and results. By combining the dense network configuration with the ReLU modification of the coupling function, we consider the following system:

$$\dot{\theta}_i(t) = \omega_i + \sum_{\ell=1}^{N} A_{i\ell} \max\{0, \sin(\theta_\ell(t) - \theta_i(t))\}, \quad \text{for } t > 0 \text{ and } i = 1, 2, \dots, N, \tag{1.6}$$

where A is the adjacency matrix defined in (1.3) without the requirement of symmetry. We then proceed to present the following results.

Theorem 1.10. *Assume $D(\Omega) = 0$. Given $\Theta(t)$ solves (1.6) with $D(\Theta(0)) < \pi$ and A is connected. Then $\Theta(t)$ achieves complete phase synchronization asymptotically. Moreover,*

$$\lim_{t\to\infty} \theta_i(t) = \max\{\theta_1(0), \theta_2(0), \dots, \theta_N(0)\}, \quad \textit{for all } i = 1, 2, \dots, N. \tag{1.7}$$

Theorem 1.11. *Let $0 < \alpha < \frac{\pi}{2}$, and suppose A is connected. Furthermore, we assume that the connectivity degree μ and the natural frequencies satisfy*

$$D(\Omega) < \mu N \sin\alpha. \tag{1.8}$$

Let $\Theta(t)$ be a solution of (1.6) with

$$D(\Theta(0)) < \pi - \alpha, \tag{1.9}$$

then

$$\lim_{t\to\infty} \dot{\theta}_i(t) = \max\{\omega_1, \omega_2, \dots, \omega_N\}, \quad \textit{for all } i = 1, 2, \dots, N. \tag{1.10}$$

This implies that the system achieves frequency synchronization, with all oscillators eventually converging to the largest natural frequency.

1.4 Social Network

In this paper, we extend the Kuramoto model by incorporating a ReLU function within a two-group structure. The two-group Kuramoto model has been extensively studied, particularly regarding attractive and repulsive interactions [15, 31, 36], single-oscillator connections between groups [8], and higher-order configurations involving three or more groups [8].

We introduce a novel social network model consisting of two groups, with the first group containing n members and the second group comprising m members:

$$\Theta(t) = (\theta_1(t), \theta_2(t), \dots, \theta_n(t)),$$
$$\Phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_m(t)).$$

In the context of family dynamics, siblings often compete for the distribution of parental resources. However, when they face collective family responsibilities, such as caring for aging parents or supporting younger family members, cooperation tends to emerge. This balance between competition and cooperation is a recurring theme in the fields of family economics, labor economics, development economics, and sociology [1,3,9,16,26,32,34]. In contrast to the intense repulsive interactions observed in other models [15,31,36], the competition within these family structures tends to be more moderate.

Our model reflects these dynamics by assuming that members within each group are strongly interconnected, resulting in significant intra-group competition. At the same time, there is an inherent attraction between members of different groups, capturing the interplay of competitive and cooperative behaviors across generational lines. This nuanced interaction is formalized in the following two-group Kuramoto model:

$$\begin{cases} \dot{\theta}_i(t) = \omega_i + \dfrac{K}{N}\left(\displaystyle\sum_{\ell=1}^{n} \max\{0, \sin(\theta_\ell - \theta_i)\} + \sum_{\ell=1}^{m} \sin(\phi_\ell - \theta_i)\right), \quad i = 1, 2, \dots, n, \\ \dot{\phi}_j(t) = \nu_j + \dfrac{K}{N}\left(\displaystyle\sum_{\ell=1}^{m} \max\{0, \sin(\phi_\ell - \phi_j)\} + \sum_{\ell=1}^{n} \sin(\theta_\ell - \phi_j)\right), \quad j = 1, 2, \dots, m, \end{cases} \tag{1.11}$$

where $N = n + m$ is the total number of oscillators, $K > 0$ is the coupling strength, and ω_i and ν_j are the natural frequencies for the oscillators θ_i and ϕ_j, respectively.

For convenience, we denote

$$\Omega_1 = (\omega_1, \omega_2, \dots, \omega_n),$$
$$\Omega_2 = (\nu_1, \nu_2, \dots, \nu_m).$$

We shall investigate the synchronization properties of this two-group Kuramoto model.

Theorem 1.12. *Let $0 < \alpha < \frac{\pi}{2}$. Suppose the coupling strength and the natural frequencies satisfy*

$$D(\Omega_1, \Omega_2) < \frac{K}{N} \min\{n, m\} \sin\alpha. \tag{1.12}$$

If $(\Theta(t), \Phi(t))$ is a solution of (1.11) with initial conditions satisfying

$$D(\Theta(0), \Phi(0)) < \pi - \alpha, \tag{1.13}$$

then

$$\lim_{t\to\infty} D(\dot{\Theta}(t), \dot{\Phi}(t)) \to 0.$$

Hence the oscillators $(\Theta(t), \Phi(t))$ achieve frequency synchronization.

This paper is organized as follows. In Section 2, we prove Theorem 1.10. Section 3 establishes the existence of the leading oscillator, which we then use to prove Theorem 1.11. In Section 4, we demonstrate the well-order lemma and prove Theorem 1.12. Finally, Section 5 provides numerical examples to illustrate our theorems and offers a comparison with the classical Kuramoto model.

2 Identical Oscillators on Dense Networks

In this section, we focus on the synchronization of the homogeneous ReLU version of the Kuramoto model on dense networks and provide a proof for Theorem 1.10. The proof is built upon the key properties of monotonicity and boundedness of the phase differences among the oscillators.

Proof of Theorem 1.10. Since $D(\Omega) = 0$, without loss of generality, we may assume

$$\omega_i = 0, \quad \text{for all } \ i = 1, 2, \ldots, N. \tag{2.1}$$

Additionally, we can rearrange the indices such that

$$\theta_1(0) \leq \theta_2(0) \leq \ldots \leq \theta_N(0). \tag{2.2}$$

Here, we assume that the initial conditions $\theta_i(0)$ for $i = 1, 2, \ldots, N$ are not all identical; otherwise, the oscillators would have already achieved complete phase synchronization asymptotically.

Define $\tilde{N}$ as the biggest index for which $\theta_i(0) < \theta_N(0)$, namely,

$$\tilde{N} := \max\{1 \leq \ell \leq N : \theta_\ell(0) < \theta_N(0)\}, \tag{2.3}$$

given that $\theta_1(0) < \theta_N(0)$, the existence of such $\tilde{N}$ is guaranteed.

Step 1: Since $\omega_i = 0$ for $i = 1, 2, \ldots, N$, $\{\theta_\ell(t)\}_{\ell=1}^N$ is uniformly bounded below by $\theta_1(0)$. We first establish that the set $\{\theta_\ell(t)\}_{\ell=1}^N$ is uniformly bounded above by $\theta_N(0)$ for $t \geq 0$. Suppose this is not the case. Let

$$\tilde{t} := \inf\{t \geq 0 : \theta_\ell(t) \geq \theta_N(0) \text{ for some } \ell = 1, 2, \ldots, \tilde{N}\} < \infty, \tag{2.4}$$

and $\tilde{j}$ denotes the corresponding oscillator. This implies that $\theta_{\tilde{j}}$ is the first oscillator among the first $\tilde{N}$ oscillators that reaches $\theta_N(0)$ at time $\tilde{t}$.

For all times $t < \tilde{t}$, it follows that $\sin(\theta_j(t) - \theta_k(t)) < 0$ for each $j = 1, 2, \ldots, \tilde{N}$ and each $k = \tilde{N} + 1, \ldots, N$. Consequently,

$$\dot{\theta}_k(t) = \sum_{\ell=\tilde{N}+1}^{N} A_{k\ell} \max\{0, \sin(\theta_\ell(t) - \theta_k(t))\} = 0, \tag{2.5}$$

for all $t < \tilde{t}$. This indicates that $\theta_k(t)$ remains at $\theta_N(0)$ until time $\tilde{t}$.

Given that $\tilde{t}$ is the first instance when any of the first $\tilde{N}$ oscillators reaches $\theta_N(0)$, there exists $\varepsilon > 0$ such that

$$\theta_{\tilde{j}}(t) \geq \theta_j(t), \quad \text{for all } t \in [\tilde{t} - \varepsilon, \tilde{t}) \text{ and for each } j = 1, 2, \ldots, \tilde{N}. \tag{2.6}$$

In the interval $[\tilde{t} - \varepsilon, \tilde{t})$, the dynamics of $\theta_{\tilde{j}}(t)$ is governed by

$$0 < \dot{\theta}_{\tilde{j}}(t) = \sum_{\ell=\tilde{N}+1}^{N} A_{\tilde{j}\ell} \max\{0, \sin(\theta_\ell(t) - \theta_{\tilde{j}}(t))\} = \sum_{\ell=\tilde{N}+1}^{N} A_{\tilde{j}\ell} \sin(\theta_\ell(0) - \theta_{\tilde{j}}(t)) = q \sin(\theta_N(0) - \theta_{\tilde{j}}(t)), \tag{2.7}$$

where we have used the fact that $1 \leq q := \sum_{\ell=\tilde{N}+1}^{N} A_{\tilde{j}\ell} \leq N - \tilde{N}$.

Solving (2.7) by separation of variables yields

$$t = \frac{\log(\cos(\frac{\theta_N(0) - \theta_{\tilde{j}}(t)}{2})) - \log(\sin(\frac{\theta_N(0) - \theta_{\tilde{j}}(t)}{2}))}{q} + C, \tag{2.8}$$

for some constant C. However, taking $t \to \tilde{t}$ results in a contradiction since the right-hand side tends to infinity and the left-hand side remains finite.

Inferring from (1.6) and (2.1), we see that $\dot{\theta}_i(t) \geq 0$, which implies that $\theta_i(t)$ is increasing. Since $\theta_i(t)$ is increasing and bounded above by $\theta_N(0)$, the completeness of the real numbers guarantees the following result.

Conclusion 2.1. *For any* $1 \leq i \leq N$, $\lim_{t\to\infty} \theta_i(t)$ *exists.*

Step 2: Next, we show that $\lim_{t\to\infty}\dot{\theta}_i(t)=0$ for all $1\le i\le N$. Under the assumption (2.1), by taking the limit of both sides of (1.6) as $t\to\infty$, we have

$$\lim_{t\to\infty}\dot{\theta}_i(t)=\sum_{\ell=1}^{N}A_{i\ell}\max\{0,\sin(\lim_{t\to\infty}\theta_\ell(t)-\lim_{t\to\infty}\theta_i(t))\},\tag{2.9}$$

which indicates the existence of $\lim_{t\to\infty}\dot{\theta}_i(t)$. Moreover, the existence of a nonzero limit would contradict Conclusion 2.1. Hence, $\lim_{t\to\infty}\dot{\theta}_i(t)=0$ exists for all $1\le i\le N$.

Step 3: Finally, we shall prove that any two connected oscillators converge to the same value. Assume oscillators θ_i and θ_j are connected, i.e., there is a path connecting θ_i and θ_j. Suppose

$$\theta_i=\theta_{V_0},\theta_{V_1},...,\theta_{V_n}=\theta_j,\tag{2.10}$$

and

$$A_{V_kV_{k+1}}=1,\quad \text{for all } k=0,1,...,n-1.\tag{2.11}$$

Fixed $1\le k\le n-1$, suppose without loss of generality that

$$\lim_{t\to\infty}\theta_{V_{k+1}}(t)\ge\lim_{t\to\infty}\theta_{V_k}(t).\tag{2.12}$$

Then,

$$\lim_{t\to\infty}\dot{\theta}_{V_k}(t)\ge\sin(\lim_{t\to\infty}\theta_{V_{k+1}}(t)-\lim_{t\to\infty}\theta_{V_k}(t)).\tag{2.13}$$

Since the left-hand side equals zero and $\theta_{V_{k+1}}(t)-\theta_{V_k}(t)<\pi$, this implies that $\lim_{t\to\infty}\theta_{V_k}(t)=\lim_{t\to\infty}\theta_{V_{k+1}}(t)$, thus

$$\lim_{t\to\infty}\theta_i(t)=\lim_{t\to\infty}\theta_{V_0}(t)=\lim_{t\to\infty}\theta_{V_1}(t)=\ldots=\lim_{t\to\infty}\theta_{V_n}(t)=\lim_{t\to\infty}\theta_{V_j}(t).\tag{2.14}$$

This proves the claim.

Therefore, all oscillators converge to a common value since the matrix A is connected. That is, $\Theta(t)$ achieves a complete phase synchronization asymptotically. On the other hand, we notice that

$$\theta_N(t)=\theta_N(0),\quad \text{for all } t\ge 0,\tag{2.15}$$

which implies (1.7). □

3 Non-identical Oscillators on Dense Networks

In this section, we provide the proof of Theorem 1.11. Without loss of generality, we assume that the natural frequencies are in the following order

$$0=\omega_1\ge\omega_2\ge\ldots\ge\omega_N.$$

We define

$$N^*:=\max\{1\le\ell\le N:\omega_\ell=0\}.\tag{3.1}$$

If $N^*=N$, the scenario reduces to the case where all frequencies are identical, and the result of Theorem 1.11 follows directly from Theorem 1.10 by an appropriate change of variables. Consequently, we shall concentrate on the case where $N^*<N$, which introduces non-identical frequencies into the system.

We first introduce several auxiliary lemmas that will be employed in the proof of Theorem 1.11.

Lemma 3.1. *Let $0<\alpha<\frac{\pi}{2}$. Suppose the coupling strength and the natural frequencies satisfy*

$$D(\Omega)<\mu N\sin\alpha.\tag{3.2}$$

If $\Theta(t)$ is a solution of (1.6) with initial conditions satisfying

$$D(\Theta(0))<\pi-\alpha,\tag{3.3}$$

then

$$D(\Theta(t))<\alpha,\quad \textit{for all } t>T:=\frac{\pi-2\alpha}{\mu N\sin\alpha-D(\Omega)}.\tag{3.4}$$

Proof. Assume that at some moment $t_0 \geq 0$, we have $\alpha < D(\Theta(t_0)) < \pi - \alpha$. Suppose $D(\Theta(t_0)) = \theta_k(t_0) - \theta_i(t_0)$ for some $k, i \in \{1, 2, \ldots, N\}$. Then,

$$\theta_i(t_0) \leq \theta_\ell(t_0) \leq \theta_k(t_0), \quad \text{for all } 1 \leq \ell \leq N.$$

Therefore,

$$\begin{aligned}
\dot{\theta}_k(t_0) - \dot{\theta}_i(t_0) &= \omega_k - \omega_i - \sum_{\ell=1}^{N} A_{i\ell} \sin(\theta_\ell(t_0) - \theta_i(t_0)) \\
&\leq D(\Omega) - \sum_{\ell=1}^{N} A_{i\ell} \sin(\theta_\ell(t_0) - \theta_i(t_0)) \\
&\leq D(\Omega) - \mu N \sin\alpha \\
&< 0. \qquad (3.5)
\end{aligned}$$

This inequality indicates that $D(\Theta(t))$ decays at a rate greater than $\mu N \sin\alpha - D(\Omega)$. This implies (3.4). □

Next, we show that there is a leading oscillator.

Lemma 3.2. *Let $\Theta(t)$ be a solution of (1.6) with initial conditions satisfying (3.2) and (3.3). Then, there exists an index $1 \leq i \leq N^*$ such that*

$$\theta_i(t) \geq \theta_j(t), \quad \text{for all } 1 \leq j \leq N \text{ and } t \geq T^* := \frac{\pi - 2\alpha}{\mu N \sin\alpha - D(\Omega)} + \frac{\alpha}{-\omega_{N^*+1}}. \qquad (3.6)$$

Proof. **Step 1:** Let T be as defined in (3.4). We shall first prove that

$$\max_{1 \leq \ell \leq N^*} \{\theta_\ell(t)\} > \max_{N^*+1 \leq \ell \leq N} \{\theta_\ell(t)\}, \quad \text{for all } t \geq T^*. \qquad (3.7)$$

To establish this, we formulate the following claim.

Claim 3.3. *If there exists some moment $t_1 \geq T$ such that $\max_{1 \leq \ell \leq N^*}\{\theta_\ell(t_1)\} - \max_{N^*+1 \leq \ell \leq N}\{\theta_\ell(t_1)\} > 0$, then it follows that*

$$\max_{1 \leq \ell \leq N^*} \{\theta_\ell(t)\} - \max_{N^*+1 \leq \ell \leq N} \{\theta_\ell(t)\} > 0, \quad \text{for all } t \geq t_1. \qquad (3.8)$$

Assume, on the contrary, that this is not the case. Let $t = t_2 > t_1$ be the first moment such that

$$\max_{1 \leq i \leq N^*} \{\theta_i(t)\} = \max_{N^*+1 \leq j \leq N} \{\theta_j(t)\}.$$

Suppose $\max_{1 \leq \ell \leq N^*}\{\theta_\ell(t_1)\} = \theta_j(t_1)$ for some $1 \leq j \leq N^*$ and $\max_{N^*+1 \leq \ell \leq N}\{\theta_\ell(t_1)\} = \theta_k(t_1)$ for some $N^* + 1 \leq k \leq N$. Then, there exists $\varepsilon > 0$ such that

$$\theta_j(t) - \theta_k(t) > 0, \quad \text{for all } t \in [t_2 - \varepsilon, t_2) \text{ and } \theta_j(t_2) = \theta_k(t_2). \qquad (3.9)$$

Consequently,

$$\dot{\theta}_j(t_2) - \dot{\theta}_k(t_2) \leq 0. \qquad (3.10)$$

On the other hand, since

$$\max_{1 \leq \ell \leq N^*} \{\theta_\ell(t_2)\} = \theta_j(t_2) = \theta_k(t_2) = \max_{N^*+1 \leq \ell \leq N} \{\theta_\ell(t_2)\}, \qquad (3.11)$$

we have

$$\sin(\theta_\ell(t_2) - \theta_j(t_2)) \leq 0, \qquad (3.12)$$
$$\sin(\theta_\ell(t_2) - \theta_k(t_2)) \leq 0, \qquad (3.13)$$

for all $1 \leq \ell \leq N$.

Subtracting the derivatives of the j-th and k-th oscillators from (1.6) yields

$$\dot{\theta}_j(t_2) - \dot{\theta}_k(t_2) = \omega_j - \omega_k = -\omega_k > 0, \tag{3.14}$$

which contradicts (3.10). This proves the claim.

Next, consider the scenario where $\max_{1\leq \ell \leq N^*}\{\theta_\ell(t)\} \leq \max_{N^*+1\leq \ell\leq N}\{\theta_\ell(t)\}$ for some $t \geq T$. Suppose that $\max_{1\leq \ell\leq N^*}\{\theta_\ell(t)\} = \theta_j$ for some $1 \leq j \leq N^*$ and $\max_{N^*+1\leq \ell\leq N}\{\theta_\ell(t)\} = \theta_k$ for some $N^* + 1 \leq k \leq N$. By Lemma 3.1, we know that $D(\Theta(t)) < \alpha$. Thus,

$$\begin{aligned} \dot{\theta}_k(t) - \dot{\theta}_j(t) &= (\omega_k - \omega_j) - \sum_{\ell=1}^{N} A_{j\ell} \max\{0, \sin(\theta_\ell(t) - \theta_j(t))\} \\ &\leq \omega_k \leq \omega_{N^*+1}. \end{aligned} \tag{3.15}$$

This observation suggests that $\max_{1\leq \ell\leq N^*}\{\theta_\ell(t)\} - \max_{N^*+1\leq \ell\leq N}\{\theta_\ell(t)\}$ decreases at a rate faster than $-\omega_{N^*+1}$. This implies (3.7).

Step 2: By (3.1), we see that $\omega_1 = \omega_2 = \ldots = \omega_{N^*} = 0$. Without loss of generality, we may assume that

$$\theta_1(T^*) = \max_{1\leq \ell\leq N^*}\{\theta_\ell(T^*)\}.$$

Using an argument analogous to that in Step 2 of Theorem 1.10, which established that $\Theta(t)$ is uniformly bounded above, we deduce that for every $1 \leq i \leq N$ the oscillator θ_i cannot reach θ_1 unless

$$\theta_i(T^*) = \theta_1(T^*) \quad \text{and} \quad \omega_i = 0.$$

This completes the proof of the lemma.

□

Given the non-differentiable nature of the max function, it becomes crucial to use its right derivative instead of the conventional derivative for analysis. The right derivative of a real-valued function f, defined over an open subset $U \subseteq \mathbb{R}$, is calculated at any point x within U as follows, provided the limit exists:

$$\partial_+ f(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}, \quad \text{for all } x \in U.$$

Building on this concept, we employ a lemma referenced in [21].

Lemma 3.4. *Consider an open set $U \subseteq \mathbb{R}$ and a set of continuous functions $f_1, f_2, \ldots, f_n$, each possessing a defined right derivative. Define $F(x) = \max_{1\leq i\leq n} f_i(x)$. It follows that the right derivative of F at any point $x \in U$ is given by*

$$\partial_+ F(x) = \max_{i\in I_x} \partial_+ f_i(x),$$

where I_x is defined as $\{i : 1 \leq i \leq n,\ F(x) = f_i(x)\}$.

Next, we prove the following key lemma, which implies Theorem 1.11.

Lemma 3.5. *Under the assumption of Theorem 1.11, we define*

$$f(t) = \max_{1\leq \ell\leq N}\{\dot{\theta}_\ell(t)\}, \tag{3.16}$$

$$g(t) = \min_{1\leq \ell\leq N}\{\dot{\theta}_\ell(t)\}. \tag{3.17}$$

Then,

$$\bar{f} := \lim_{t\to\infty} f(t) = \bar{g} := \lim_{t\to\infty} g(t) = \omega_1 = \max\{\omega_1, \omega_2, \ldots, \omega_N\}, \tag{3.18}$$

which implies that

$$\lim_{t\to\infty} \dot{\theta}_i(t) = \omega_1 = \max\{\omega_1, \omega_2, \ldots, \omega_N\}, \quad \textit{for all } i = 1, 2, \ldots, N. \tag{3.19}$$

Proof. **Step 1.** We show that $\bar{f}$ exists in this step. Let T^* be defined in (3.6). For any fixed $t \geq T^*$, suppose $f(t) = \dot{\theta}_i(t)$ for some $1 \leq i \leq N$.

Inferring from Lemma 3.4,

$$\begin{aligned}
&\partial_+ \max\{0, \sin(\theta_\ell(t) - \theta_i(t))\} \\
&= \begin{cases} \cos(\theta_\ell(t) - \theta_i(t)) \left(\dot{\theta}_\ell(t) - \dot{\theta}_i(t)\right) & \text{if } \sin(\theta_\ell(t) - \theta_i(t)) > 0; \\ \max\left\{0, \cos(\theta_\ell(t) - \theta_i(t)) \left(\dot{\theta}_\ell(t) - \dot{\theta}_i(t)\right)\right\} & \text{if } \sin(\theta_\ell(t) - \theta_i(t)) = 0; \\ 0 & \text{if } \sin(\theta_\ell(t) - \theta_i(t)) < 0. \end{cases}
\end{aligned} \tag{3.20}$$

From Lemma 3.1, $\cos(\theta_\ell(t) - \theta_i(t)) > \cos\alpha > 0$. Also, by the definition of i, $\left(\dot{\theta}_\ell(t) - \dot{\theta}_i(t)\right) \leq 0$ for all $1 \leq \ell \leq N$.

Hence we may conclude that,

$$\partial_+ f(t) = \sum_{\ell=1}^N A_{i\ell} \partial_+ \max\{0, \sin(\theta_\ell(t) - \theta_i(t))\} \leq 0. \tag{3.21}$$

Additionally, $f(t)$ is bounded below by ω_N. Hence, by the completeness of the real numbers, $\bar{f}$ exists. Similarly, $\bar{g}$ exists.

Step 2: We assert that if $\bar{f} = 0$, then $\bar{g} = 0$. For the sake of contradiction, assume that $\bar{g} \neq 0$. Since $\bar{f} = 0$, it follows that $\bar{g} < 0$. By the continuity of the functions f and g, there exists a moment $T' > 0$ such that for all $t \geq T'$ the following inequalities hold:

$$0 \leq f(t) \leq -\frac{\bar{g}}{2(N-2)}, \tag{3.22}$$

$$2\bar{g} \leq g(t) \leq \bar{g}. \tag{3.23}$$

Then, for any $t \geq \max\{T', T^*\}$, we have

$$\sum_{\ell=2}^N \dot{\theta}_\ell(t) \leq g(t) + (N-2) f(t) \leq \bar{g} - \frac{\bar{g}}{2} \leq \frac{\bar{g}}{2}. \tag{3.24}$$

Integrating (3.24) over the interval

$$\left[\max\{T', T^*\},\ \max\{T', T^*\} - \frac{2(N-2)\alpha}{\bar{g}}\right],$$

we obtain

$$\sum_{\ell=2}^N \left(\theta_\ell\left(\max\{T', T^*\} - \frac{2(N-2)\alpha}{\bar{g}}\right) - \theta_\ell\left(\max\{T', T^*\}\right)\right) = \int_{\max\{T', T^*\}}^{\max\{T', T^*\} - \frac{2(N-2)\alpha}{\bar{g}}} \sum_{\ell=2}^N \dot{\theta}_\ell(t)\, dt \leq -(N-2)\alpha. \tag{3.25}$$

By the pigeonhole principle, there exists at least one oscillator, say $\theta_{\tilde{\ell}}(t)$, for which

$$\left|\theta_{\tilde{\ell}}\left(\max\{T', T^*\} - \frac{2(N-2)\alpha}{\bar{g}}\right) - \theta_{\tilde{\ell}}\left(\max\{T', T^*\}\right)\right| \geq \alpha. \tag{3.26}$$

However, by Lemma 3.2, there exists a leading oscillator θ_1 with natural frequency zero, implying that its phase remains constant for all $t \geq T^*$.

By applying the triangle inequality in conjunction with (3.26), we deduce that either

$$D\Big(\Theta\big(\max\{T', T^*\}\big)\Big) \geq \theta_1(\max\{T', T^*\}) - \theta_{\tilde{\ell}}(\max\{T', T^*\}) \geq \alpha \quad \text{or} \tag{3.27}$$

$$\begin{aligned}
D\Big(\Theta\Big(\max\{T', T^*\} - \frac{2(N-2)\alpha}{\bar{g}}\Big)\Big) &\geq \theta_1(\max\{T', T^*\} - \frac{2(N-2)\alpha}{\bar{g}}) - \theta_{\tilde{\ell}}(\max\{T', T^*\} - \frac{2(N-2)\alpha}{\bar{g}}) \\
&\geq \alpha.
\end{aligned} \tag{3.28}$$

which contradicts Lemma 3.1. Therefore, our assumption is false, and we conclude that $\bar{g} = 0$.

Step 3: In this step, we consider that $\bar{f} > 0$ and $t \geq T^*$.

Claim 3.6. *If $\dot\theta_i(t) > 0$ for some index i, then the set*

$$C_i(t) := \{\, 1 \le \ell \le N : \theta_\ell(t) > \theta_i(t) \text{ and } A_{i\ell} = 1 \,\} \tag{3.29}$$

is nonempty.

Indeed, since

$$0 < \dot\theta_i(t) = \omega_i + \sum_{\ell=1}^{N} A_{i\ell} \max\{0, \sin(\theta_\ell(t) - \theta_i(t))\}, \tag{3.30}$$

and $\omega_i \le 0$, we conclude that

$$\sum_{\ell=1}^{N} A_{i\ell} \max\{0, \sin(\theta_\ell(t) - \theta_i(t))\} > 0, \tag{3.31}$$

which implies $C_i(t) \ne \emptyset$.

To reach a contradiction, our strategy is to use Claim 3.6 to identify a sequence of oscillators whose phases increase while their frequencies remain positive.

Define $a_1(t)$ by

$$f(t) = \theta_{a_1(t)}(t). \tag{3.32}$$

If multiple indices satisfy this equality, we select the oscillator that, over the subsequent short time interval, continues to have the highest frequency; by continuity, such a choice is well defined. Similarly, if the same situation recurs later, the same selection rule is applied. Next, define

$$b_2(t) = \min\Big\{\dot\theta_{a_1(t)}(t) - \dot\theta_\ell(t) : \ell \in C_{a_1(t)}(t)\Big\}, \tag{3.33}$$

$$a_2(t) = \arg\min\Big\{\dot\theta_{a_1(t)}(t) - \dot\theta_\ell(t) : \ell \in C_{a_1(t)}(t)\Big\}. \tag{3.34}$$

Since $\theta_{a_1(t)}(t)$ has the maximum frequency, $b_2(t)$ is nonnegative. Moreover, by the selection of $a_2(t)$ and the continuity of $\dot\theta_i(t)$ for $1 \le i \le N$, it follows that $b_2(t)$ is right continuous.

By Lemma 3.1 and (3.20), we deduce that

$$\begin{aligned} \partial_+ f(t) &\le A_{a_1(t)a_2(t)}\, \partial_+ \max\Big\{0,\, \sin\big(\theta_{a_2(t)}(t) - \theta_{a_1(t)}(t)\big)\Big\} \\ &\le -\cos\alpha b_2(t). \end{aligned} \tag{3.35}$$

Thus, $f(t)$ decreases at a rate of at least $\cos\alpha b_2(t)$. Since $\bar f$ exists and $b_2(t)$ is right continuous, it follows that $\cos\alpha b_2(t)$ is integrable. Moreover, because $b_2(t)$ is nonnegative, the set

$$I_1 := \Big\{t \ge T^* : b_2(t) \ge \frac{\bar f}{N}\Big\} \tag{3.36}$$

has finite measure. We also define

$$J_1 := \Big\{t \ge T^* : b_2(t) < \frac{\bar f}{N}\Big\}. \tag{3.37}$$

Then,

$$\dot\theta_{a_2(t)}(t) = f(t) - b_2(t) \ge \bar f - \frac{\bar f}{N} > 0 \quad \text{for all } t \in J_1. \tag{3.38}$$

By Claim 3.6, $C_{a_2(t)}(t)$ is nonempty.

Therefore, we may define

$$b_3(t) = \min\Big\{\dot\theta_{a_2(t)}(t) - \dot\theta_\ell(t) : \ell \in C_{a_2(t)}(t)\Big\}, \tag{3.39}$$

$$a_3(t) = \arg\min\Big\{\dot\theta_{a_2(t)}(t) - \dot\theta_\ell(t) : \ell \in C_{a_2(t)}(t)\Big\}. \tag{3.40}$$

Similarly, $b_3(t)$ is right continuous. As is customary, we define the positive and negative parts of $b_3(t)$ by

$$b_3^+(t) = \max\{0,\, b_3(t)\}, \tag{3.41}$$
$$b_3^-(t) = \max\{0,\, -b_3(t)\}, \tag{3.42}$$

so that

$$b_3(t) = b_3^+(t) - b_3^-(t). \tag{3.43}$$

Claim 3.7. *Both $b_3^+(t)$ and $b_3^-(t)$ are integrable; hence, $b_3(t)$ is integrable as well.*

Since $\theta_{a_1(t)}(t)$ has the maximum frequency, we have

$$\dot\theta_{a_1(t)}(t) - b_2(t) - b_3^+(t) + b_3^-(t) = \dot\theta_{a_1(t)}(t) - b_2(t) - b_3(t) = \dot\theta_{a_3(t)}(t) \le \dot\theta_{a_1(t)}(t). \tag{1010}$$

This implies that

$$0 \le b_3^-(t) \le b_2(t). \tag{3.44}$$

Since $b_2(t)$ is integrable and $b_3(t)$ is right continuous, it follows that $b_3^-(t)$ is integrable too.

From (1.6), we obtain

$$|\dot\theta_i(t)| \le M := N - \min_{1\le i\le N} \omega_i \quad \text{for all } t > 0 \text{ and } i = 1, 2, \ldots, N. \tag{3.45}$$

Hence,

$$|\dot\theta_i(t) - \dot\theta_j(t)| \le 2M, \quad \text{for all } t > 0 \text{ and } 1 \le i, j \le N. \tag{3.46}$$

In particular,

$$b_3(t) \le 2M, \quad \text{for all } t. \tag{3.47}$$

Partition the set $\{t : t \ge T^*\}$ as

$$\{t : t \ge T^*\} = S_1 \cup S_2 \cup S_3, \tag{3.48}$$

where

$$S_1 = I_1, \tag{3.49}$$
$$S_2 = J_1 \cap \{t : b_3(t) \le 0\}, \tag{3.50}$$
$$S_3 = J_1 \cap \{t : b_3(t) > 0\}. \tag{3.51}$$

Combining with (1.6), (3.20) and (3.46), for $t \in S_1$ we have

$$|\partial_+ \dot\theta_{a_2(t)}(t)| \le NM. \tag{3.52}$$

For $t \in S_2$, $\theta_{a_2(t)}(t)$ is increasing and

$$|\partial_+ \dot\theta_{a_2(t)}(t)| \le N b_3^-(t). \tag{3.53}$$

For $t \in S_3$, $\theta_{a_2(t)}(t)$ is decreasing and

$$|\partial_+ \dot\theta_{a_2(t)}(t)| \ge \cos\alpha b_3^+(t). \tag{3.54}$$

Suppose that

$$\int_0^\infty \cos\alpha\, b_3^+(t)\, dt = \infty. \tag{3.55}$$

Then, there exists $T_0 > T^*$ such that

$$\int_{T^*}^{T_0} \cos\alpha\, b_3^+(t)\,dt > \frac{NM\int_0^\infty 1_{S_1}(t)\,dt + \int_0^\infty N\,b_3^-(t)\,dt + 2M}{\cos\alpha}. \tag{3.56}$$

By combining (3.52), (3.53), (3.54) and (3.56), we have

$$\begin{aligned}
\dot\theta_{a_2}(T_0) - \dot\theta_{a_2}(T^*) &= \int_{T^*}^{T_0} \partial_+\dot\theta_{a_2(t)}(t)\,dt \\
&\le \int_{T^*}^{T_0} NM1_{S_1}(t)\,dt + \int_{T^*}^{T_0} N\,b_3^-(t)1_{S_2}(t)\,dt - \int_{T^*}^{T_0} \cos\alpha\, b_3^+(t)1_{S_2}(t)\,dt \\
&< NM\int_0^\infty 1_{S_1}(t)\,dt + \int_0^\infty N\,b_3^-(t)\,dt - \cos\alpha \times \frac{NM\int_0^\infty 1_{S_1}(t)\,dt + \int_0^\infty N\,b_3^-(t)\,dt + 2M}{\cos\alpha} \\
&= -2M,
\end{aligned} \tag{3.57}$$

where $1_{S_i}(t)$ denotes the indicator function of S_i, this contradicts (3.45).

Therefore, we must have

$$\int_0^\infty \cos\alpha\, b_3^+(t)\,dt < \infty, \tag{3.58}$$

which completes the proof of the claim.

Now, we define

$$I_2 := \Big\{t \ge T^* : b_3(t) \ge \frac{\bar f}{N}\Big\}, \tag{3.59}$$

$$J_2 := \Big\{t \ge T^* : b_3(t) < \frac{\bar f}{N}\Big\}. \tag{3.60}$$

By Claim 3.7, the measure of I_2 is finite, which implies that $J_1 \cap J_2$ is nonempty. Therefore, according to the definitions of J_1 and J_2,

$$\dot\theta_{a_3(t)}(t) = f(t) - b_2(t) - b_3(t) > 0, \quad \text{for all } t \in J_1 \cap J_2. \tag{3.61}$$

By iterating the above procedure, we can construct $(N+1)$ distinct oscillators for some t, each with a positive frequency. This, however, is impossible because there are only N oscillators. Hence, we obtain the contradiction $\bar f > 0$, which forces $\bar f = 0$.

Conclusion 3.8. *Since the maximum frequency and the minimum frequency converge to ω_1, it follows that all the frequencies converge to ω_1. Consequently, frequency synchronization occurs, thereby proving Theorem 1.11.*

□

4 Social Network

In this section, we focus on the social network model described by (1.11) and proceed with the proof of Theorem 1.12. We begin with several auxiliary lemmas.

Lemma 4.1. *Let $0 < \alpha < \frac{\pi}{2}$. Suppose the coupling strength and the natural frequencies satisfy*

$$D(\Omega_1, \Omega_2) < \frac{K}{N}\min\{n, m\}\sin\alpha. \tag{4.1}$$

If $(\Theta(t), \Phi(t))$ is a solution of (1.11) with initial conditions satisfying

$$D(\Theta(0), \Phi(0)) < \pi - \alpha, \tag{4.2}$$

then

$$D(\Theta(t), \Phi(t)) < \pi - \alpha, \quad \textit{for all } t > 0. \tag{4.3}$$

Moreover, there exists a positive number T such that

$$D(\Theta(t),\Phi(t)) < \alpha \quad \textit{for all } \ t > T := \frac{\pi - 2\alpha}{\frac{K}{N}\min\{n,m\}\sin\alpha - D(\Omega_1,\Omega_2)}. \tag{4.4}$$

Proof. **Step 1**: We prove (4.2) in this step. To do so, we employ a proof by contradiction. Let t_0 denote the first moment at which $D(\Theta(t),\Phi(t))$ attains the value $\pi-\alpha$. We consider two cases.

Case 1: Suppose $D(\Theta(t_0),\Phi(t_0)) = \theta_k(t_0) - \theta_i(t_0)$ for some $k,i \in \{1,2,\dots,n\}$ (the symmetric case $D(\Theta(t_0),\Phi(t_0)) = \phi_k(t_0)-\phi_i(t_0)$ for some $k,i\in\{1,2,\dots,m\}$ is similar). Clearly,

$$\dot\theta_k(t_0) - \dot\theta_i(t_0) \geq 0. \tag{3.5}$$

Utilizing equation (1.11), we derive

$$\begin{aligned}
\dot\theta_k(t_0) - \dot\theta_i(t_0) &= \omega_k - \omega_i + \frac{K}{N}\sum_{\ell=1}^{n}\left(\max\{0,\sin(\theta_\ell-\theta_k)\} - \max\{0,\sin(\theta_\ell-\theta_i)\}\right)\\
&\quad + \frac{K}{N}\sum_{\ell=1}^{m}\left(\sin(\phi_\ell-\theta_k) - \sin(\phi_\ell-\theta_i)\right)\\
&= \omega_k - \omega_i + \frac{K}{N}\sum_{\ell=1}^{n}\left(\max\{0,\sin(\theta_\ell-\theta_k)\} - \max\{0,\sin(\theta_\ell-\theta_i)\}\right) && (4.5)\\
&\quad - \frac{2K}{N}\sin\left(\frac{\theta_k-\theta_i}{2}\right)\sum_{\ell=1}^{m}\cos\left(\phi_\ell - \frac{\theta_k+\theta_i}{2}\right). && (4.6)
\end{aligned}$$

Given the definitions of θ_k and θ_i, for any $1 \leq \ell \leq m$,

$$\phi_\ell(t_0) - \frac{\theta_k(t_0)+\theta_i(t_0)}{2} \leq \frac{\theta_k(t_0)-\theta_i(t_0)}{2} = \frac{\pi-\alpha}{2}. \tag{4.7}$$

Thus,

$$\cos\left(\phi_\ell(t_0) - \frac{\theta_k(t_0)+\theta_i(t_0)}{2}\right) \geq \cos\left(\frac{\pi-\alpha}{2}\right) > 0. \tag{4.8}$$

Furthermore, as $D(\Theta(t),\Phi(t)) < \pi-\alpha$ for all $t \leq t_0$, and with θ_k being the oscillator with the largest phase at t_0, $\sin(\theta_\ell-\theta_k) \leq 0$ for all $\ell \in \{1,2,\dots,n\}$, hence $\max\{0,\sin(\theta_\ell-\theta_k)\} = 0$. At $t = t_0$, applying (4.1), we find

$$\begin{aligned}
\dot\theta_k(t_0) - \dot\theta_i(t_0) &\leq D(\Omega_1,\Omega_2) - \frac{2K}{N}m\sin\left(\frac{\pi-\alpha}{2}\right)\cos\left(\frac{\pi-\alpha}{2}\right) && (4.9)\\
&= D(\Omega_1,\Omega_2) - \frac{K}{N}m\sin\alpha < 0, && (4.10)
\end{aligned}$$

which leads to a contradiction.

Case 2: Suppose $D(\Theta(t_0),\Phi(t_0)) = \theta_k(t_0)-\phi_i(t_0)$ for some $k\in\{1,2,\dots,n\}$ and $i\in\{1,2,\dots,m\}$ (the case $D(\Theta(t_0),\Phi(t_0)) = \phi_k(t_0)-\theta_i(t_0)$ for some $k\in\{1,2,\dots,m\}$ and $i\in\{1,2,\dots,n\}$ is similar). By the argument in (4.8), we have

$$\begin{aligned}
\dot\theta_k(t_0) - \dot\phi_i(t_0) &= \omega_k - \nu_i + \frac{K}{N}\sum_{\ell=1}^{n}\left(\max\{0,\sin(\theta_\ell-\theta_k)\} - \sin(\theta_\ell-\phi_i)\right) && (4.11)\\
&\quad + \frac{K}{N}\sum_{\ell=1}^{m}\left(\sin(\phi_\ell-\theta_k) - \max\{0,\sin(\phi_\ell-\phi_i)\}\right) && (4.12)\\
&\leq D(\Omega_1,\Omega_2) + \frac{K}{N}\sum_{\ell=1}^{m}\left(\sin(\phi_\ell-\theta_k) - \sin(\phi_\ell-\phi_i)\right) && (4.13)\\
&\leq D(\Omega_1,\Omega_2) - \frac{K}{N}m\sin\alpha < 0, && (4.14)
\end{aligned}$$

which leads to a contradiction.

Step 2: For (4.4), suppose $D(\Theta(t_0), \Phi(t_0)) = \theta_k(t_0) - \theta_i(t_0) \geq \alpha$ for some $k, i \in \{1, 2, \ldots, n\}$ (the same logic applies to the other cases). From (4.9), we have

$$\dot{\theta}_k(t_0) - \dot{\theta}_i(t_0) \leq D(\Omega_1, \Omega_2) - \frac{K}{N} m \sin\alpha < 0. \tag{4.15}$$

This means $D(\Theta(t), \Phi(t))$ decreases at a rate faster than $\frac{K}{N} m \sin\alpha - D(\Omega_1, \Omega_2)$. If $D(\Theta(t_0), \Phi(t_0)) = \theta_k(t_0) - \theta_i(t_0) \geq \alpha$, it implies that

$$D(\Theta(t), \Phi(t)) < \alpha, \quad \text{for } t > T_1 := \frac{\pi - 2\alpha}{\frac{K}{N} m \sin\alpha - D(\Omega_1, \Omega_2)}. \tag{4.16}$$

□

Lemma 4.2. *Let $(\Theta(t), \Phi(t))$ satisfy the dynamics described by (1.11). Assume there exists t_0 such that $D(\Theta(t), \Phi(t)) < \alpha$ for all $t > t_0$. If $\omega_k > \omega_i$, then there exists $T_{k,i}$ such that*

$$\theta_k(t) > \theta_i(t), \quad \textit{for all}\, t > T_{k,i}. \tag{4.17}$$

If $\omega_k = \omega_i$, then there exists a time $T_{k,i}$ such that one of the following holds:

$$\theta_k(t) > \theta_i(t), \quad \textit{for all}\, t > T_{k,i}, \tag{4.18}$$

$$\theta_k(t) = \theta_i(t), \quad \textit{for all}\, t > T_{k,i}, \tag{4.19}$$

$$\theta_k(t) < \theta_i(t), \quad \textit{for all}\, t > T_{k,i}. \tag{4.20}$$

An analogous assertion holds for $\Phi(t)$, and we use $G_{k,i}$ to denote the counterpart of $T_{k,i}$.

Proof. We first consider the case where $\omega_k > \omega_i$. Suppose $\theta_k(t) > \theta_i(t)$ at any $t_1 > t_0$. We claim that $\theta_k(t) > \theta_i(t)$ for all $t \geq t_1$. To prove by contradiction, assume there exists $t' > t_1$ where $\theta_k(t') = \theta_i(t')$, marking the first instance of equality. Consequently,

$$\dot{\theta}_k(t') - \dot{\theta}_i(t') \leq 0. \tag{4.21}$$

However, subtracting the derivatives of the k-th and i-th oscillators from (1.11) yields

$$\dot{\theta}_k(t') - \dot{\theta}_i(t') = \omega_k - \omega_i > 0, \tag{4.22}$$

which is a contradiction, we then verify our claim.

Next, if $\theta_k(t_0) - \theta_i(t_0) > 0$, the case is already settled. Otherwise, if $\theta_k(t_0) - \theta_i(t_0) \leq 0$, we find

$$\begin{aligned} \dot{\theta}_i(t) - \dot{\theta}_k(t) = (\omega_i - \omega_k) + \frac{K}{N}\left(\sum_{\ell=1}^{m} \max\{0, \sin(\phi_\ell - \phi_i)\} - \max\{0, \sin(\phi_\ell - \phi_k)\}\right) \\ + \frac{K}{N}\left(\sum_{\ell=1}^{n} \sin(\theta_\ell - \phi_i) - \sum_{\ell=1}^{n} \sin(\theta_\ell - \phi_k)\right). \end{aligned} \tag{4.23}$$

Given $D(\Theta(t), \Phi(t)) < \alpha < \frac{\pi}{2}$,

$$\dot{\theta}_i(t) - \dot{\theta}_k(t) \leq \omega_i - \omega_k < 0, \tag{4.24}$$

confirming that $\theta_k(t) - \theta_i(t)$ increases faster than $\omega_k - \omega_i$. Thus, we can define

$$T_{k,i} = t_0 + \max\left\{0, \frac{\theta_k(t_0) - \theta_i(t_0)}{\omega_k - \omega_i}\right\}.$$

For the case where $\omega_k = \omega_i$, consider two subcases. If $\theta_k(t_0) = \theta_i(t_0)$, then applying (1.11), we have

$$\theta_k(t) = \theta_i(t), \quad \text{for all } t \geq T_{k,i} := t_0. \tag{4.25}$$

Otherwise, without loss of generality, assume $\theta_k(t_0) > \theta_i(t_0)$. If (4.18) does not hold for $T_{k,i} := t_0$, then there must exist a time t_1 such that $\theta_k(t_1) = \theta_i(t_1)$. According to (1.11) again, this implies that $\theta_k(t) = \theta_i(t)$ for all $t \geq T_{k,i} := t_1$, thus establishing statement (4.20).

An analogous assertion holds for $\Phi(t)$. This completes the proof. □

Building upon the results of Lemmas 4.1 and 4.2, we derive the subsequent lemma.

Lemma 4.3 (Well-ordering Lemma)**.** *Let $(\Theta(t), \Phi(t))$ be governed by (1.11). Assuming (4.1) and (4.2) are satisfied, there exists $T^* > 0$ such that*

- *$D(\Theta(t), \Phi(t)) < \alpha$ for all $t > T$.*
- *If $\omega_k > \omega_i$, then*

$$\theta_k(t) > \theta_i(t), \quad \text{for all } t > T^*. \tag{4.26}$$

- *If $\omega_k = \omega_i$, then one of the following holds:*

$$\theta_k(t) > \theta_i(t), \quad \text{for all } t > T^*, \tag{4.27}$$

$$\theta_k(t) = \theta_i(t), \quad \text{for all } t > T^*, \tag{4.28}$$

$$\theta_k(t) < \theta_i(t), \quad \text{for all } t > T^*. \tag{4.29}$$

An analogous well-order property holds for $\Phi(t)$ with respect to this T^*.

Proof. Assume t_0 is the time T established in Lemma 4.1. Applying Lemma 4.2, we consider every pair $(\theta_k(t), \theta_i(t))$ where $\omega_k \geq \omega_i$, and each pair $(\phi_k(t), \phi_i(t))$ where $\nu_k \geq \nu_i$. We then determine T^* as

$$T^* := \max\left(\max_{(k,i):\omega_k \geq \omega_i} T_{k,i}, \max_{(k,i):\nu_k \geq \nu_i} G_{k,i}\right) \tag{4.30}$$

to validate the lemma, where $T_{k,i}$ and $G_{k,i}$ are defined in Lemma 4.2. □

We are now ready to derive Theorem 1.12, as a direct result of the following lemma.

Lemma 4.4. *Consider the system governed by (1.11) with initial conditions satisfying (4.1) and (4.2). Define the functions*

$$f(t) = \max_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \{\dot{\theta}_i(t), \dot{\phi}_j(t)\}, \tag{4.31}$$

$$g(t) = \min_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \{\dot{\theta}_i(t), \dot{\phi}_j(t)\}. \tag{4.32}$$

Then,

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t), \tag{4.33}$$

which implies that $\lim_{t\to\infty} D(\dot{\Theta}(t), \dot{\Phi}(t)) = 0$.

Proof. Assume, without loss of generality, that

$$\omega_1 \leq \omega_2 \leq \ldots \leq \omega_n,$$
$$\nu_1 \leq \nu_2 \leq \ldots \leq \nu_m.$$

Step 1: We first establish the existence of $\lim_{t\to\infty} f(t)$.

Let T^* be as defined in (4.30). For any fixed $t > T^*$, suppose $f(t) = \dot{\theta}_i(t)$ for some $1 \leq i \leq n$. (The case where $f(t) = \dot{\phi}_j(t)$ for some $1 \leq j \leq m$ is analogous.) By Lemma 4.3, there exists $1 \leq i' \leq n$ such that

$$\theta_\ell(t') \leq \theta_i(t'), \quad \text{for all } t' > T^* \text{ and } 1 \leq \ell \leq i', \tag{4.34}$$

$$\theta_\ell(t') > \theta_i(t'), \quad \text{for all } t' > T^* \text{ and } i' < \ell \leq n. \tag{4.35}$$

Thus, we can express $\dot{\theta}_i(t)$ as

$$\begin{aligned}\dot{\theta}_i(t) &= \omega_i + \frac{K}{N}\left(\sum_{\ell=1}^{n} \max\{0, \sin(\theta_\ell - \theta_i)\} + \sum_{\ell=1}^{m} \sin(\phi_\ell - \theta_i)\right) \\ &= \omega_i + \frac{K}{N}\left(\sum_{\ell=i'+1}^{n} \sin(\theta_\ell - \theta_i) + \sum_{\ell=1}^{m} \sin(\phi_\ell - \theta_i)\right). \end{aligned} \tag{4.36}$$

Differentiating $f(t)$ with respect to t, we obtain

$$f'(t) = \frac{K}{N}\left(\sum_{\ell=i'+1}^{n}\cos(\theta_\ell-\theta_i)(\dot\theta_\ell(t)-\dot\theta_i(t)) + \sum_{\ell=1}^{m}\cos(\phi_\ell-\theta_i)(\dot\phi_\ell(t)-\dot\theta_i(t))\right). \tag{4.37}$$

By Lemma 4.1, both $|\theta_\ell-\theta_i|$ and $|\phi_\ell-\theta_i|$ are less than $\alpha<\frac{\pi}{2}$. Therefore, $\cos(\theta_\ell-\theta_i)$ and $\cos(\phi_\ell-\theta_i)$ are greater than $\cos(\alpha)>0$. Additionally, since oscillator i has the highest frequency, both $(\dot\theta_\ell(t)-\dot\theta_i(t))$ and $(\dot\phi_\ell(t)-\dot\theta_i(t))$ are less than or equal to zero. Consequently, $f'(t)\le 0$. Moreover, by (1.11), $f(t)$ is bounded below by

$$\min(\Omega_1,\Omega_2)-K.$$

Thus, by the completeness of the real numbers, $\lim_{t\to\infty} f(t)$ exists. Similarly, $g(t)$ is monotone increasing to $\lim_{t\to\infty} g(t)$.

Step 2: Next, we prove that $\lim_{t\to\infty} f(t)=\lim_{t\to\infty} g(t)$. Suppose, for the sake of contradiction, that

$$\bar f := \lim_{t\to\infty} f(t) > \lim_{t\to\infty} g(t) = \bar g. \tag{4.38}$$

By the monotonicity of f and g, we can choose T_0 such that

$$\bar f + \frac{\bar f-\bar g}{2} \ge f(t) \ge \bar f, \quad \text{for all } t>T_0, \tag{4.39}$$

$$\bar g - \frac{\bar f-\bar g}{2} \le g(t) \le \bar g, \quad \text{for all } t>T_0. \tag{4.40}$$

Subtracting (4.40) from (4.39) yields

$$\bar f-\bar g \le f(t)-g(t) \le 2(\bar f-\bar g), \quad \text{for all } t>T_0. \tag{4.41}$$

Fix $t>\max\{T^*,T_0\}$. If $g(t)=\phi_j(t)$ for some $1\le j\le m$, then by the argument in (4.37), we have

$$\begin{aligned} f'(t)-g'(t) &\le f'(t) \\ &\le \frac{K}{N}\cos(\alpha)(\dot\phi_j(t)-\dot\theta_i(t)) \\ &\le \frac{K}{N}\cos(\alpha)(\bar g-\bar f). \end{aligned} \tag{4.42}$$

On the other hand, if $g(t)=\theta_j(t)$ for some $1\le j\le n$, then by the argument in (4.37) again, we have

$$\begin{aligned} f'(t)-g'(t) &\le \frac{K}{N}\cos(\alpha)\Big((\dot\phi_1(t)-\dot\theta_i(t))-(\dot\phi_1(t)-\dot\theta_j(t))\Big) \\ &= \frac{K}{N}\cos(\alpha)(\dot\theta_j(t)-\dot\theta_i(t)) \\ &\le \frac{K}{N}\cos(\alpha)(\bar g-\bar f). \end{aligned} \tag{4.43}$$

In both cases, $f(t)-g(t)$ decreases at a rate faster than $\frac{K}{N}\cos(\alpha)(\bar f-\bar g)$.

Setting $t=\max\{T^*,T_0\}+\frac{3}{2}\cdot\frac{1}{\frac{K}{N}\cos(\alpha)}$ yields a contradiction to (4.41). Therefore, we conclude that $\bar f=\bar g$, ensuring phase synchronization. □

5 Numerical Results

In this section, we present numerical simulations for both the ReLU variant of the Kuramoto model on dense networks, described by (1.6), and the ReLU variant of the social network model, given by (1.11). For the dense networks, we also include a comparative analysis with the synchronization behavior observed in the classical Kuramoto model (1.4). Additionally, we provide an illustrative example of a scenario that does not meet the conditions outlined in our theorems, yet still achieves synchronization.

5.1 Identical Oscillators on Dense Networks

As stated in Theorem 1.10, when $D(\Theta(0)) < \pi$ and the adjacency matrix A is connected, $\Theta(t)$ achieves complete phase synchronization. To illustrate this, we use numerical results (see Figure 1). Specifically, we examine the coupling behavior of 10 oscillators. The initial positions of the oscillators are uniformly generated from the interval $[0, \frac{5}{6}\pi]$, with $\theta_1(0) = 0$ and $\theta_{10}(0) = \frac{5}{6}\pi$ to ensure that $D(\Theta(0)) = \frac{5}{6}\pi$. The connections between oscillators are shown in Figure 1a. As guaranteed by Theorem 1.10, $\Theta(t)$ achieves complete phase synchronization.

Next, we compare the results of the ReLU version (1.6) with those of the classical Kuramoto model (1.4). Given that $\mu = \frac{8}{9} > 0.75$, and based on the findings of Kassabov et al. [24], the classical Kuramoto model (1.4) also achieves complete phase synchronization. As shown in Figure 2, the classical model converges faster than the ReLU version.

5.2 Non-identical Oscillators on Dense Networks

In this subsection, we investigate the case of non-identical oscillators, with parameters set to $N = 10$, $\mu = \frac{8}{9}$, and $\alpha = \frac{\pi}{3}$. The initial natural frequencies and phases are specified as

$$\begin{aligned}\Omega &= (0.250, 0.138, 0.088, 0.076, 0.044, -0.005, -0.031, -0.167, -0.193, -0.200),\\ \Theta(0) &= (3.902, 4.905, 4.454, 4.427, 5.609, 3.572, 5.518, 5.429, 5.498, 4.081).\end{aligned}$$

The network topology connecting the oscillators is depicted in Figure 3a.

Hence the initial conditions meet the following criteria:

$$\begin{aligned}D(\Omega) &\approx 0.450 \le 7.698 \approx \mu N \sin\alpha,\\ D(\Theta(0)) &\approx 2.036 \le 2.094 \approx \pi - \alpha,\\ \max_{1\le i\le 10}(|\omega_i|) &\approx 0.250 \le 0.262 \approx \sqrt{\mu - \frac{3}{4}} + \mu - 1.\end{aligned}$$

Consequently, the conditions of Theorem 1.5 and Theorem 1.11 are satisfied, and their conclusions can be applied to this scenario.

Figure 3 illustrates the synchronization outcomes, whereas Figure 4 compares these results with those obtained using the classical Kuramoto model. Consistent with the observations in Figure 1, the classical model achieves frequency synchronization more rapidly than the ReLU-modified version.

5.3 Social Network

We simulate the interaction between two groups of oscillators as described by the social network model (1.11). Group 1 consists of $n = 2$ members, while Group 2 has $m = 3$ members, resulting in a total of $N = 5$ oscillators. The coupling strength is set to $K = 1$, and the angle α is set to $\frac{\pi}{4}$.

The initial natural frequencies and phases for the groups are

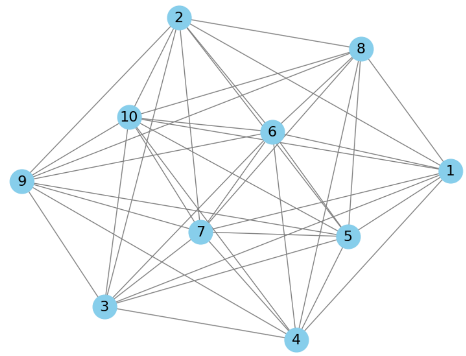
$$\begin{aligned}\Omega_1 &= (0, 0.108),\\ \Omega_2 &= (0.078, 0.247, 0.264),\\ \Theta(0) &= (3.103, 4.103),\\ \Phi(0) &= (5.026, 3.099, 2.864).\end{aligned}$$

These initial conditions ensure that $D(\Omega_1, \Omega_2) = 0.264$, which is less than $\frac{2}{5}\sin(\frac{\pi}{4}) \approx 0.283$, and $D(\Theta(0), \Phi(0)) = 2.162$, which is less than $\pi - \frac{\pi}{4} \approx 2.356$. Thus, the requirements of Theorem (1.12) are satisfied.

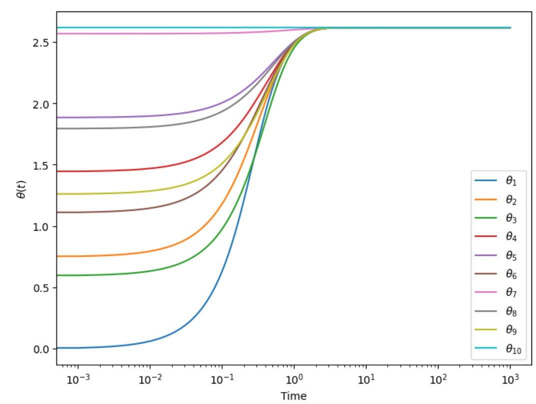
Figure 5 presents the results of our simulation. These results demonstrate that frequency synchronization is achieved, which aligns with our theoretical predictions.

5.4 Synchronization Beyond Our Theorem

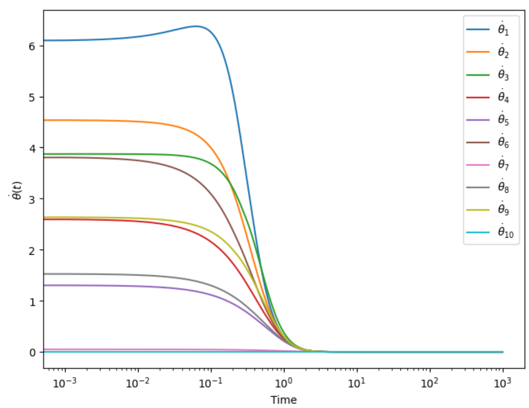
In this section, we present two sets of numerical simulations: one demonstrating synchronization under conditions that do not strictly satisfy our theoretical results, and another exploring the dynamics within a social network model.



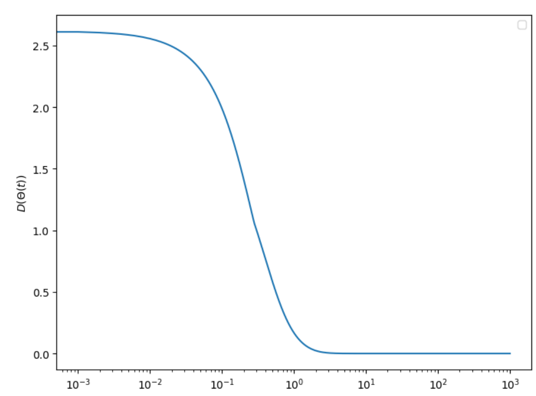
(a) Visualization of the connections between oscillators.



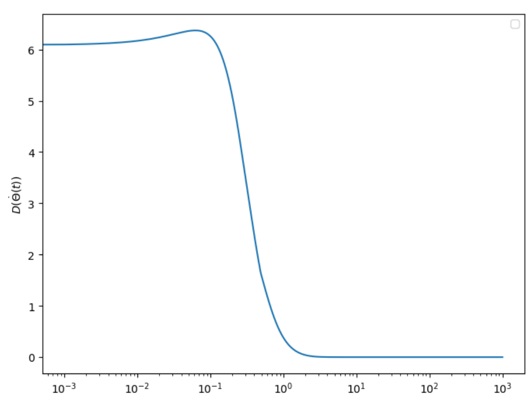
(b) Evolution of phases $\Theta(t)$.



(c) Evolution of frequency $\dot{\Theta}(t)$.

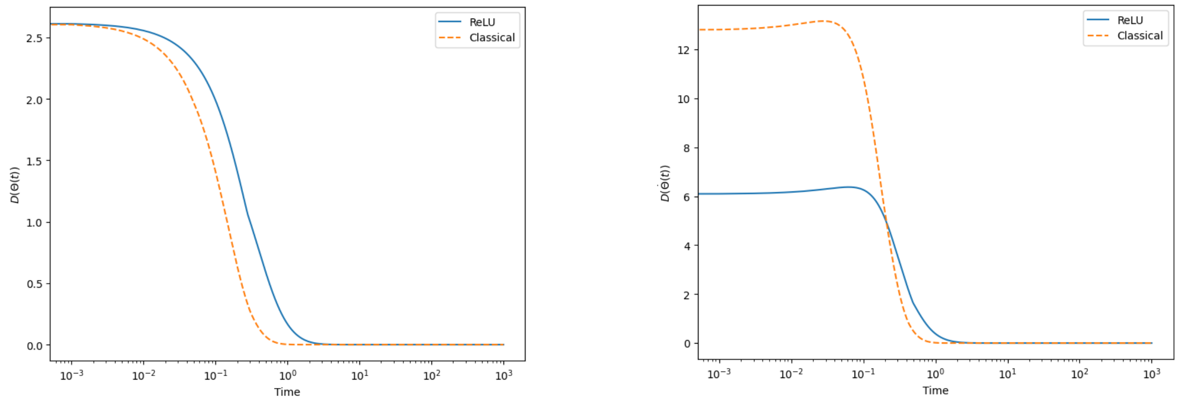


(d) Diameter of phase $D(\Theta(t))$.



(e) Diameter of frequency $D(\dot{\Theta}(t))$.

Figure 1: Solution of the ReLU version of the Kuramoto model on dense networks (1.6) with $N = 10$, $D(\Omega) = 0$, $D(\Theta(0)) = \frac{5}{6}\pi$, $\mu = \frac{8}{9}$, and the connection is visualized as in Figure 1a.



(a) Diameter of phase $D(\Theta(t))$.

(b) Diameter of frequency $D(\dot{\Theta}(t))$.

Figure 2: Comparison of the ReLU version of the homogeneous Kuramoto model (1.6) and the classical homogeneous Kuramoto model (1.4).

First, we consider a case where the system achieves synchronization despite not fulfilling the conditions specified in our theorems. Following a similar setup to that in Section 5.1, we examine the behavior of 10 oscillators with initial phases uniformly distributed over the interval $[0, \frac{7}{6}\pi]$. The phases are set such that $\theta_1(0) = 0$ and $\theta_{10}(0) = \frac{7}{6}\pi$, ensuring that the initial phase difference is $D(\Theta(0)) = \frac{15}{8}\pi$. The connectivity structure of the oscillators is depicted in Figure 6a.

The results, presented in Figure 6, reveal that the system achieves both phase and frequency synchronization, even with a relatively low connectivity parameter $\mu = \frac{2}{9}$.

Next, we explore the non-identical case, where the parameters are set to $N = 10$ and $\mu = \frac{2}{9}$. The initial natural frequencies and phases of the oscillators are given by:

$$\Omega = (0, -0.098, -0.182, -0.360, -0.369, -0.390, -0.524, -0.593, -0.635, -0.984),$$
$$\Theta(0) = (2.098, 2.176, 2.564, 2.348, 1.071, 1.296, 2.579, 0.876, 2.528, 2.312).$$

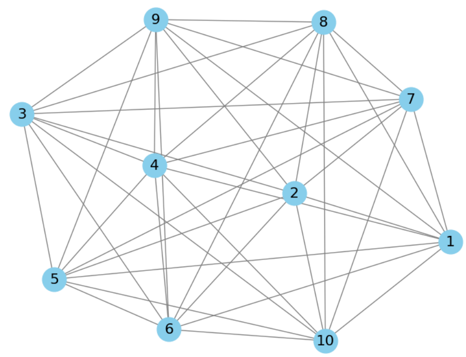
The network structure is shown in Figure 7a.

Under these conditions, $D(\Omega) = 0.984$ exceeds the connectivity parameter $\mu = \frac{2}{9}$, indicating that the system does not meet the criteria outlined in Theorem 1.11. Nevertheless, as illustrated in Figure 7, the oscillators still achieve frequency synchronization.

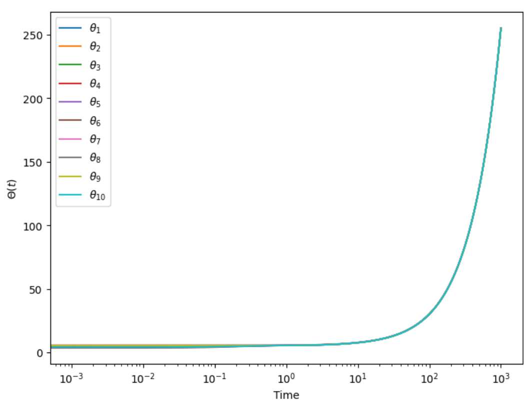
Finally, we examine the social network model described by (1.11), focusing on the interaction between two groups of oscillators. Group 1 comprises $n = 2$ members, while Group 2 consists of $m = 3$ members, for a total of $N = 5$ oscillators. The initial natural frequencies and phases are specified as follows:

$$\Omega_1 = (0, 0.028),$$
$$\Omega_2 = (0.783, 0.725, 0.334),$$
$$\Theta(0) = (1.693, 1.494),$$
$$\Phi(0) = (0.750, 0.296, 0.393).$$

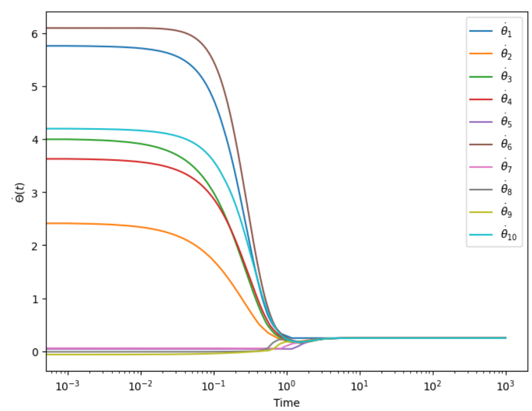
In this case, $D(\Omega_1, \Omega_2) = 0.783$ exceeds the connectivity threshold $\mu = \frac{2}{5}$, meaning that the system does not satisfy the requirements of Theorem 1.12. However, as shown in Figure 8, the phases $(\Theta(t), \Phi(t))$ still achieve frequency synchronization, illustrating the robustness of the model even under less favorable conditions.



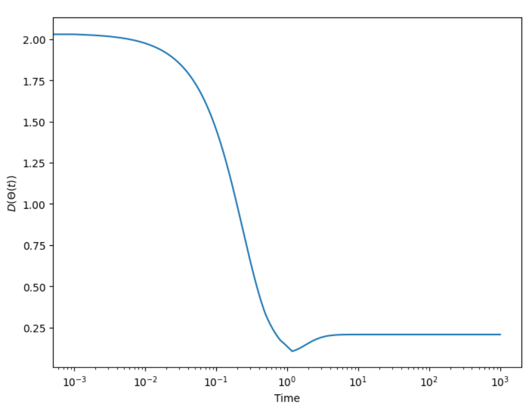
(a) Visualization of the connections between oscillators.



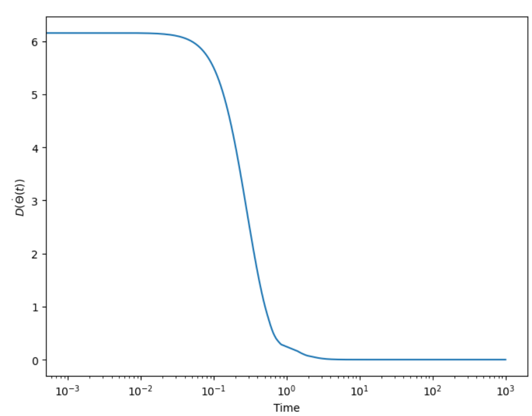
(b) Evolution of phases $\Theta(t)$.



(c) Evolution of frequency $\dot{\Theta}(t)$.

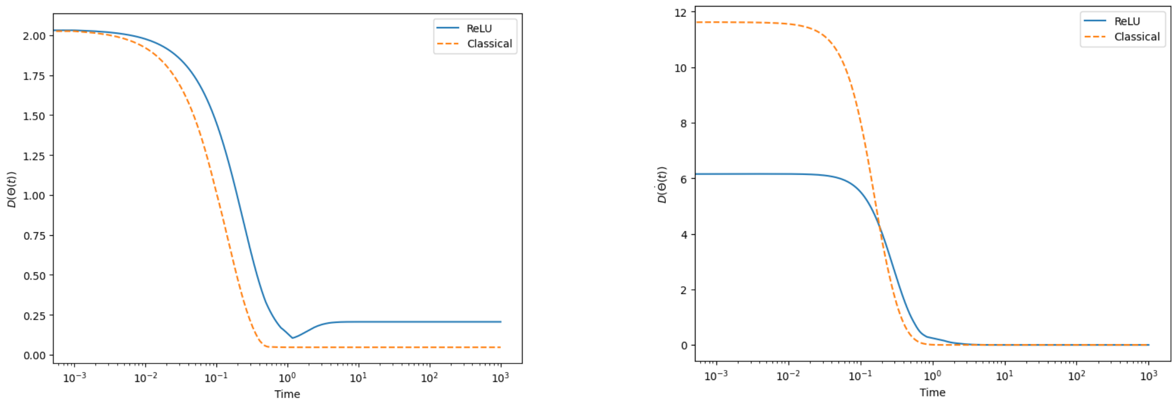


(d) Diameter of phase $D(\Theta(t))$.



(e) Diameter of frequency $D(\dot{\Theta}(t))$.

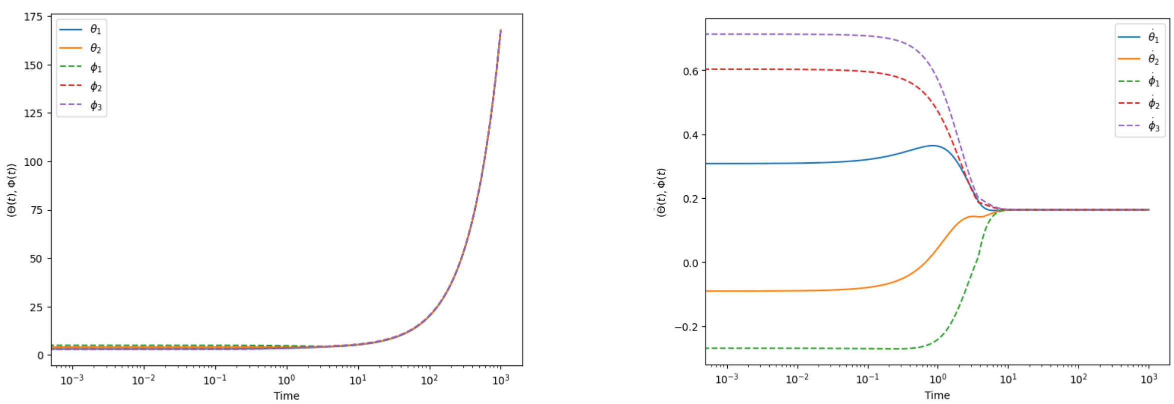
Figure 3: Solution of the ReLU version of the nonhomogeneous Kuramoto model on dense networks (1.4) with $N = 10$, $\alpha = \frac{\pi}{3}$ and $\mu = \frac{8}{9}$, and the connection is visualized as in Figure 3a.



(a) Diameter of phase $D(\Theta(t))$.

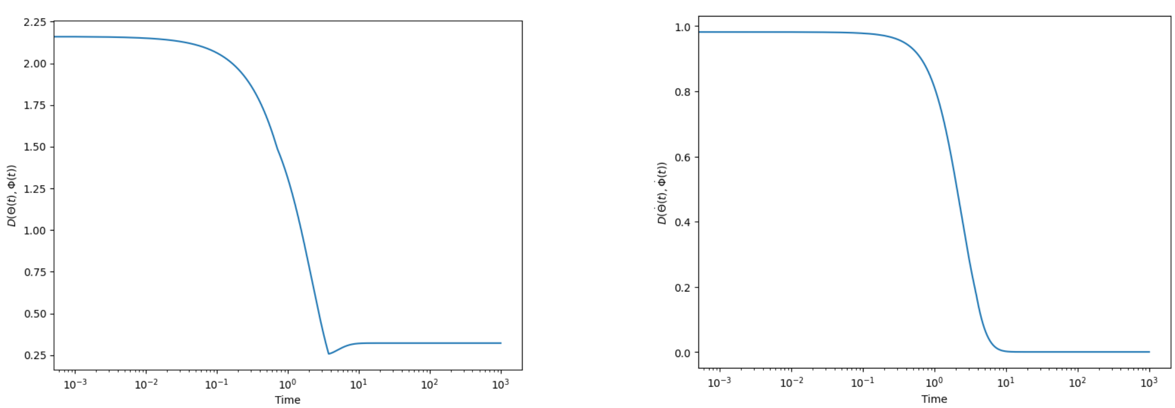
(b) Diameter of frequency $D(\dot{\Theta}(t))$.

Figure 4: Comparison of the ReLU version of the nonhomogeneous Kuramoto model (1.6) and the classical nonhomogeneous Kuramoto model (1.4).



(a) Evolution of phases $(\Theta(t), \Phi(t))$.

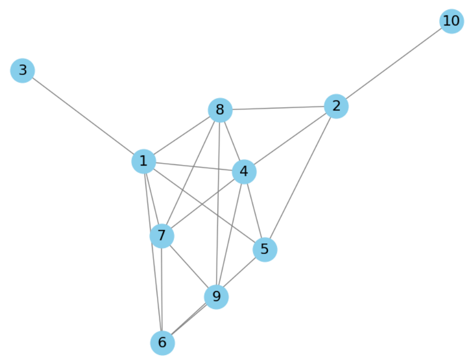
(b) Evolution of frequency $(\dot{\Theta}(t), \dot{\Phi}(t))$.



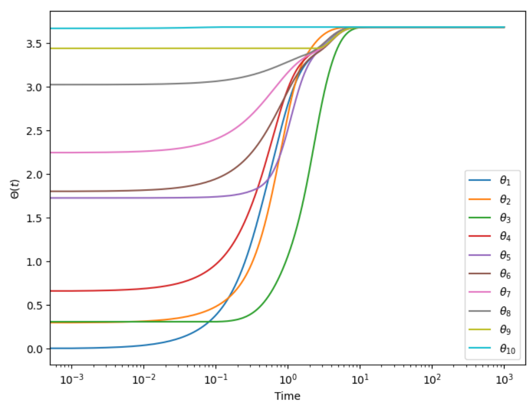
(c) Diameter of phases $D(\Theta(t), \Phi(t))$.

(d) Diameter of frequency $D(\dot{\Theta}(t), \dot{\Phi}(t))$.

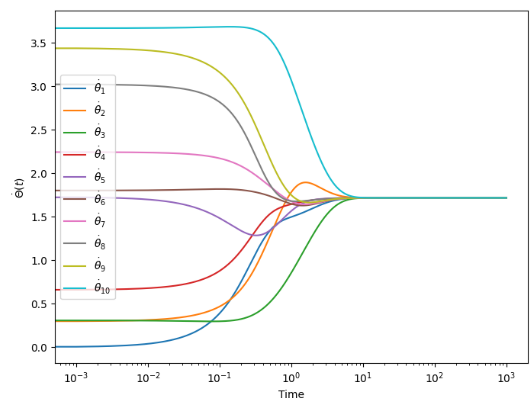
Figure 5: Solution of (1.11) with $n = 2$, $m = 3$, $N = 5$, $K = 1$, $\alpha = \frac{\pi}{4}$.



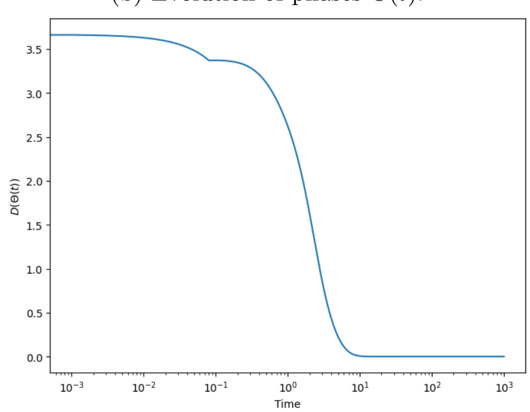
(a) Visualization of the connections between oscillators.



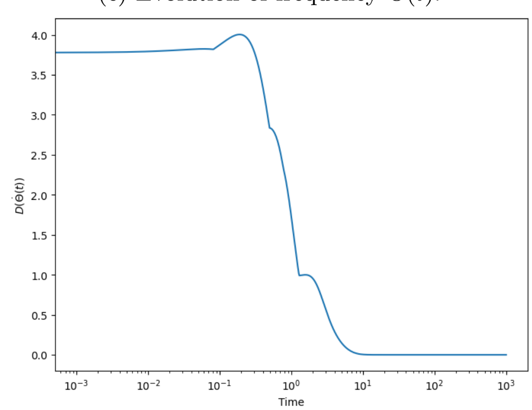
(b) Evolution of phases $\Theta(t)$.



(c) Evolution of frequency $\dot{\Theta}(t)$.

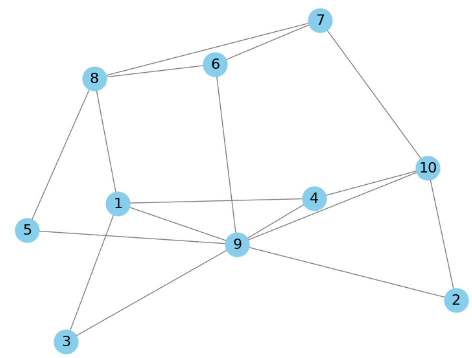


(d) Diameter of phase $D(\Theta(t))$.

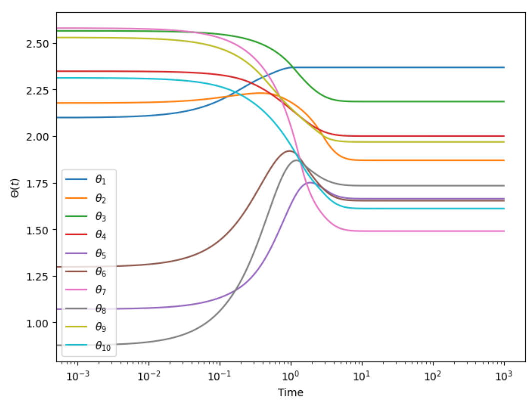


(e) Diameter of frequency $D(\dot{\Theta}(t))$.

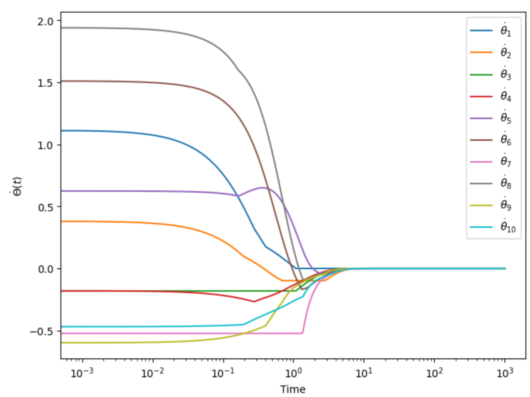
Figure 6: Solution of the ReLU version of the Kuramoto model on dense networks (1.6) with $N = 10$, $D(\Omega) = 0$, $D(\Theta(0)) = \frac{7}{6}\pi$, $\mu = \frac{1}{9}$, and the connection is visualized as in Figure 6a.



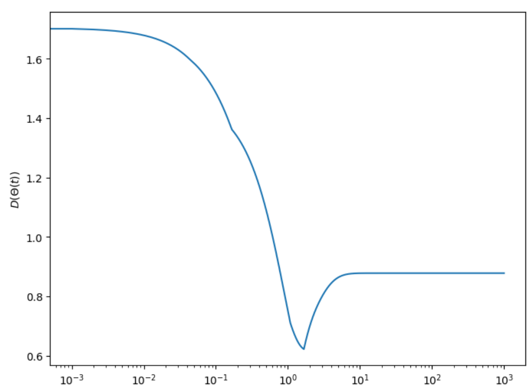
(a) Visualization of the connections between oscillators.



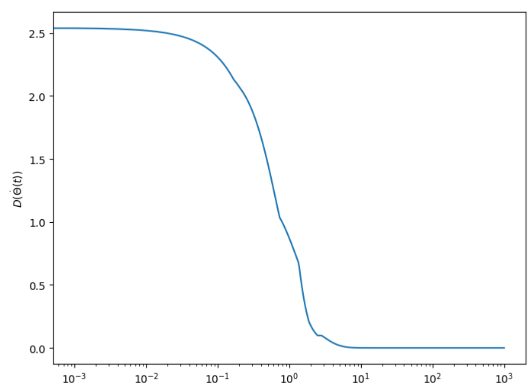
(b) Evolution of phases $\Theta(t)$.



(c) Evolution of frequency $\dot{\Theta}(t)$.

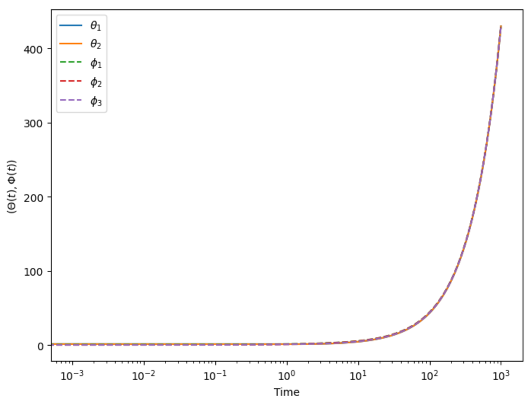


(d) Diameter of phase $D(\Theta(t))$.

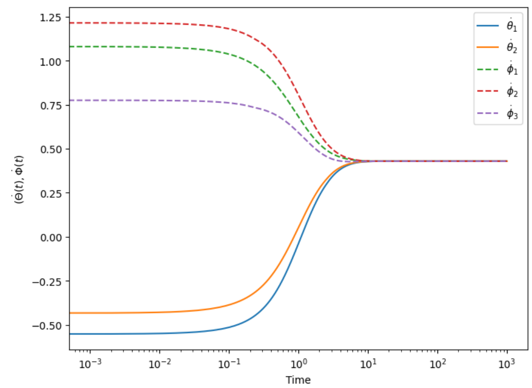


(e) Diameter of frequency $D(\dot{\Theta}(t))$.

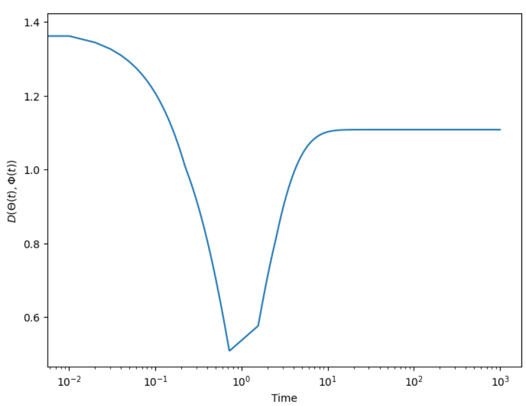
Figure 7: Solution of the ReLU version of the nonhomogeneous Kuramoto model on dense networks (1.4) with $N = 10$ and $\mu = \frac{2}{9}$, and the connection is visualized as in Figure 7a.



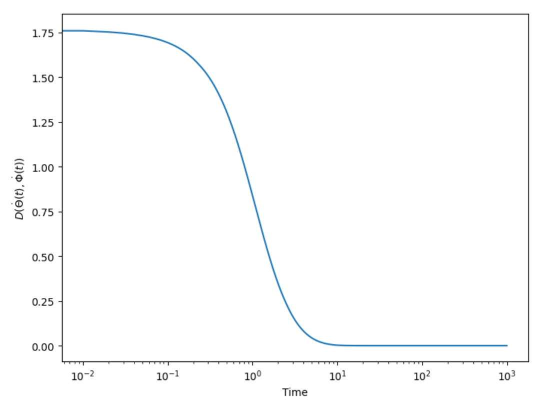
(a) Evolution of phases $(\Theta(t), \Phi(t))$.



(b) Evolution of frequency $(\dot{\Theta}(t), \dot{\Phi}(t))$.



(c) Diameter of phases $D(\Theta(t), \Phi(t))$.



(d) Diameter of frequency $D(\dot{\Theta}(t), \dot{\Phi}(t))$.

Figure 8: Solution of (1.11) with $n = 2$, $m = 3$, $N = 5$, $K = 1$.

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